Math 155, *Lecture Notes- Bonds* **Name____________**

Section 9.8 *Power Series*

In this section, we will learn that several types of important functions can be represented *exactly* by infinite series called **power series**.
For example,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$$

Eventually, we will see that for each real number x, the infinite series on the right side will converge to the number e^x.

Definition of Power Series

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a **power series.** More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

is called a **power series centered at c,** where c is a constant.

Ex. 1: (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1} =$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^3} =$

We can view $f(x)=\sum_{n=0}^{\infty} a_n (x-c)^n$ as a function of x where the domain of f is the set of all x for which the power series converges. Therefore, we will need to know the values of x that allow the series to converge, and determination of this domain will be the main focus of this section.

First, every power series converges at its center c:

$$f(c)=\sum_{n=0}^{\infty} a_n (c-c)^n$$
$$= a_0(1)+0+0+0\cdots$$, where we are agreeing that $(x-c)^0=1$, even if $x=c$.
$$= a_0$$

The domain of a power series has only three basic forms: a single point, an interval centered at c, or the entire real line.

THEOREM 9.20 Convergence of a Power Series

For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.
2. There exists a real number $R > 0$ such that the series converges absolutely for $|x - c| < R$, and diverges for $|x - c| > R$.
3. The series converges absolutely for all x.

The number R is the **radius of convergence** of the power series. If the series converges only at c, the radius of convergence is $R = 0$, and if the series converges for all x, the radius of convergence is $R = \infty$. The set of all values of x for which the power series converges is the **interval of convergence** of the power series.

Ex. 2: Determine the interval of convergence of the series: $\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$

Note that for a power series with a radius of convergence that is a finite number R, Theorem 9.20 says nothing about the convergence at the *endpoints* of the interval of convergence. In fact, each endpoint must be tested separately for convergence or divergence.

Ex. 3: Determine the interval of convergence of the series: $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

More Ex. 3:

Still More Ex. 3:

Ex. 4: Determine the interval of convergence of the series: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n}$

More Ex. 4:

Still More Ex. 4:

THEOREM 9.21 Properties of Functions Defined by Power Series

If the function given by

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$
$$= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

has a radius of convergence of $R > 0$, then, on the interval $(c-R, c+R)$, f is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of f are as follows.

1. $$f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$$
$$= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$$

2. $$\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$$
$$= C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots$$

The *radius of convergence* of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The *interval of convergence*, however, may differ as a result of the behavior at the endpoints.

Ex. 5: Determine the interval of convergence for **(a)** $f(x)$, **(b)** $f'(x)$, **(c)** $f''(x)$, and **(d)** $\int f(x)dx$, given that $f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$.

More Ex. 5:

Still More Ex. 5:

Ex. 6: Given $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$.

(a) Find the interval of convergence.

(b) Show that $f'(x)=f(x)$.

(c) Show that $f(0)=1$.

(d) Identify the function.

More Ex. 6:

More Ex. 6: